AERODYNAMICS OF LIFT GENERATION

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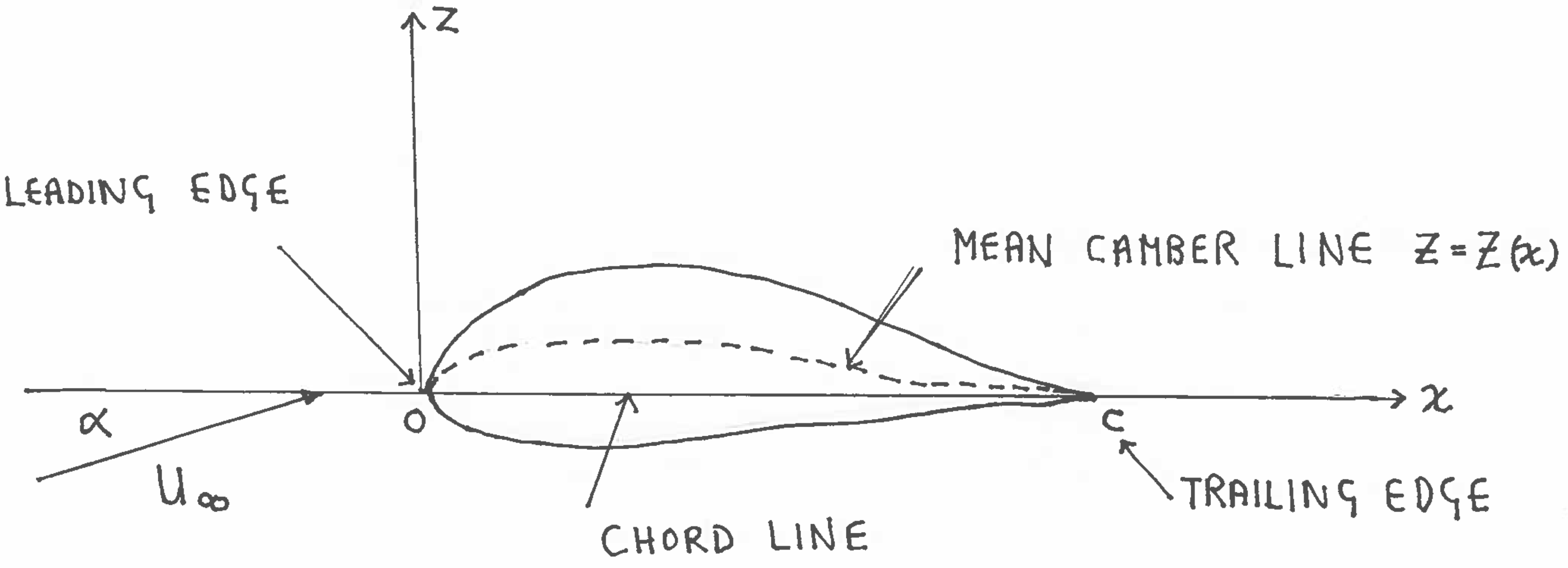
1. AEROFOIL EQUATION



- MEAN CAMBER LINE : LOCUS OF POINTS HALF WAY BETWEEN UPPER SURFACE AND LOWER SURFACE
- CHORD LINE : STRAIGHT LINE CONNECTING LEADING/TRAILING EDGES

$$\frac{1}{2\pi} \fint_0^c \frac{\gamma(\xi)\, d\xi}{\xi - x} = \left(\alpha - \frac{dz}{dx} \right) U_\infty \qquad 0 \leqslant x \leqslant c$$

VORTEX STRENGTH $\gamma(x)$ $0 \leqslant x \leqslant c$

KUTTA CONDITION $\gamma(c) = 0$

CIRCULATION $\Gamma = \int_0^c \gamma(x)\, dx$ $(\Gamma < 0)$

LIFT $L = -\rho U_\infty \Gamma$

CAUCHY PRINCIPAL VALUE INTEGRAL

J. ANDERSON, FUNDAMENTALS OF AERODYNAMICS

" THE SOLUTION OF THE AEROFOIL EQUATION CAN BE OBTAINED FROM THE MATHEMATICAL THEORY OF INTEGRAL EQUATIONS, WHICH IS BEYOND THE SCOPE OF THIS BOOK".

TRANSFORMATION: $\eta = \xi - \frac{c}{2}$, $x^* = x - \frac{c}{2}$

$\eta = \frac{c}{2} \cos\theta$ $x^* = \frac{c}{2} \cos\phi$

$$\frac{1}{2\pi} \int_0^{\pi} \frac{\gamma(\theta) \sin\theta \, d\theta}{\cos\theta - \cos\phi} = \left(\alpha - \frac{dz}{dx^*} \right) U_\infty$$

SYMMETRIC AEROFOIL

CAMBER ZERO $Z(x) = 0$

EFFECT OF CAMBER TREATED BY STUDENTS

EXPANSION

$$\gamma(\theta) = \frac{1}{\sin\theta}\sum_{n=0}^{\infty} \gamma_n \cos n\theta \, , \qquad \gamma_n = \text{CONSTANT}$$

AEROFOIL EQUATION

$$\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\cos n\theta \, d\theta}{\cos\theta - \cos\phi} = \alpha U_\infty$$

SOLUTION DEPENDS ON RESULT

$$\boxed{\int_0^{\infty}\frac{\cos n\theta \, d\theta}{\cos\theta - \cos\phi} = \frac{\pi \sin n\phi}{\sin\phi}}$$

WE ESTABLISH THIS RESULT

SOLVE AEROFOIL EQUATION

2. MATHEMATICAL MODEL

FLUID INVISCID INCOMPRESSIBLE

FLOW IRROTATIONAL

- COMPLEX POTENTIAL

$$W(z) = \phi(x,y) + i\,\psi(x,y) \qquad z = x + iy$$

VELOCITY POTENTIAL ϕ: $V_x = \frac{\partial \phi}{\partial x}$, $V_y = \frac{\partial \phi}{\partial y}$

STREAM FUNCTION ψ : $V_x = \frac{\partial \psi}{\partial y}$, $V_y = -\frac{\partial \psi}{\partial x}$

CAUCHY - RIEMANN EQUATIONS

$$V_x: \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$W(z)$ IS DIFFERENTIABLE

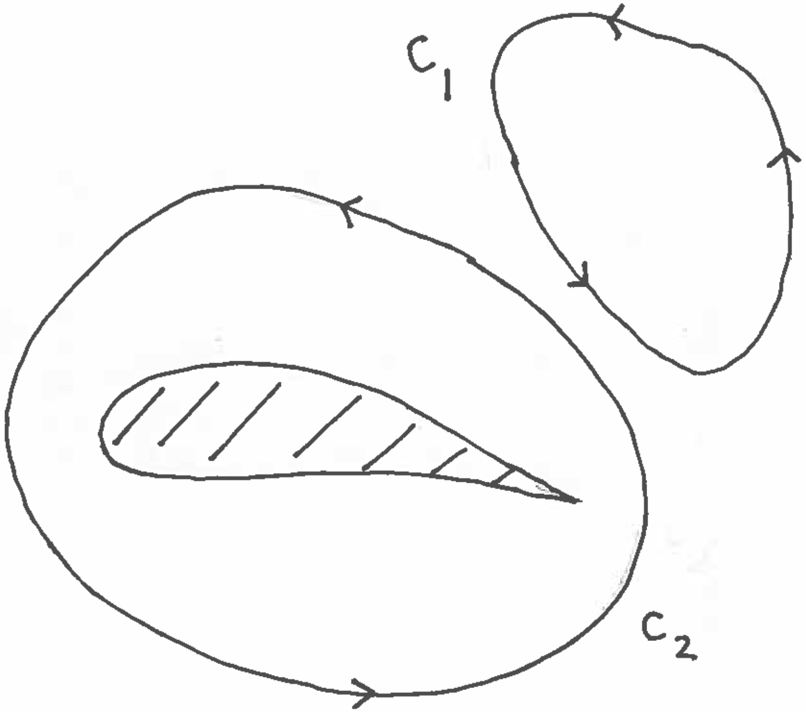
- CIRCULATION

$$\Gamma = \oint_C \underline{v} \cdot d\underline{r}$$

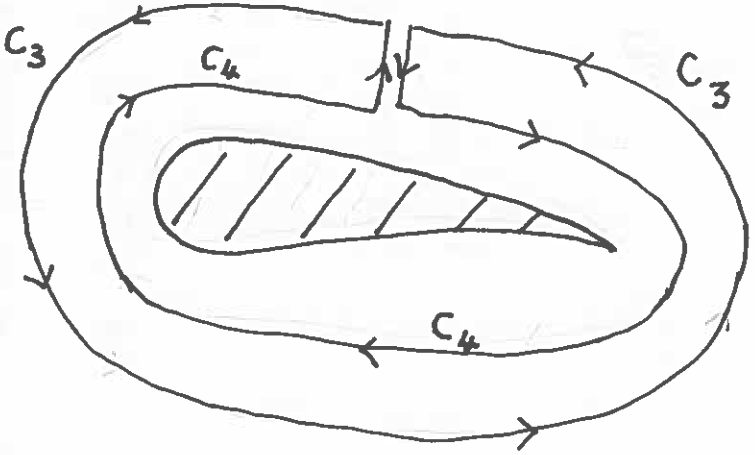
C CLOSED CURVE

$$\Gamma = \int_S (\underline{\nabla} \times \underline{v}) \cdot \underline{n} \, dS$$

$\Gamma_1 = 0$, $\Gamma_2 \neq 0$



ALL CIRCUITS C_2 HAVE SAME VALUE Γ



$\Gamma_3 - \Gamma_4 = 0$

$\Gamma_3 = \Gamma_4$

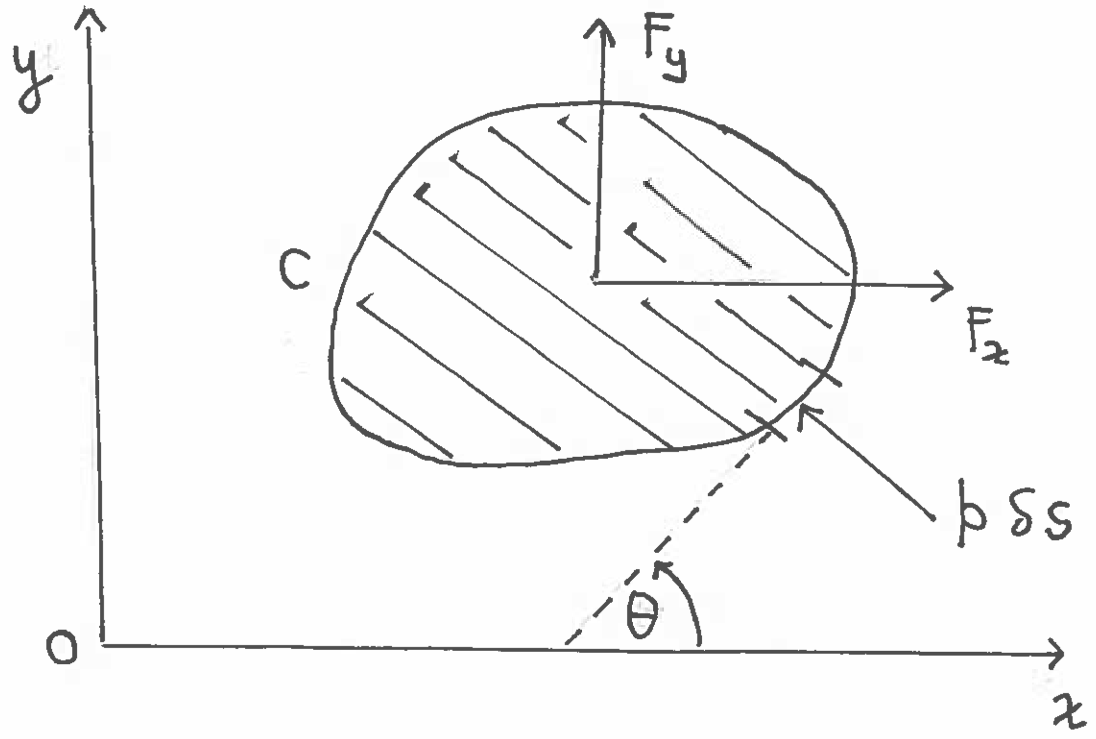
$$\frac{D\Gamma_c}{Dt} = 0$$

C CONSISTS AT ALL TIMES OF THE SAME FLUID PARTICLES.

STARTING VORTEX

- THEOREM OF BLASIUS

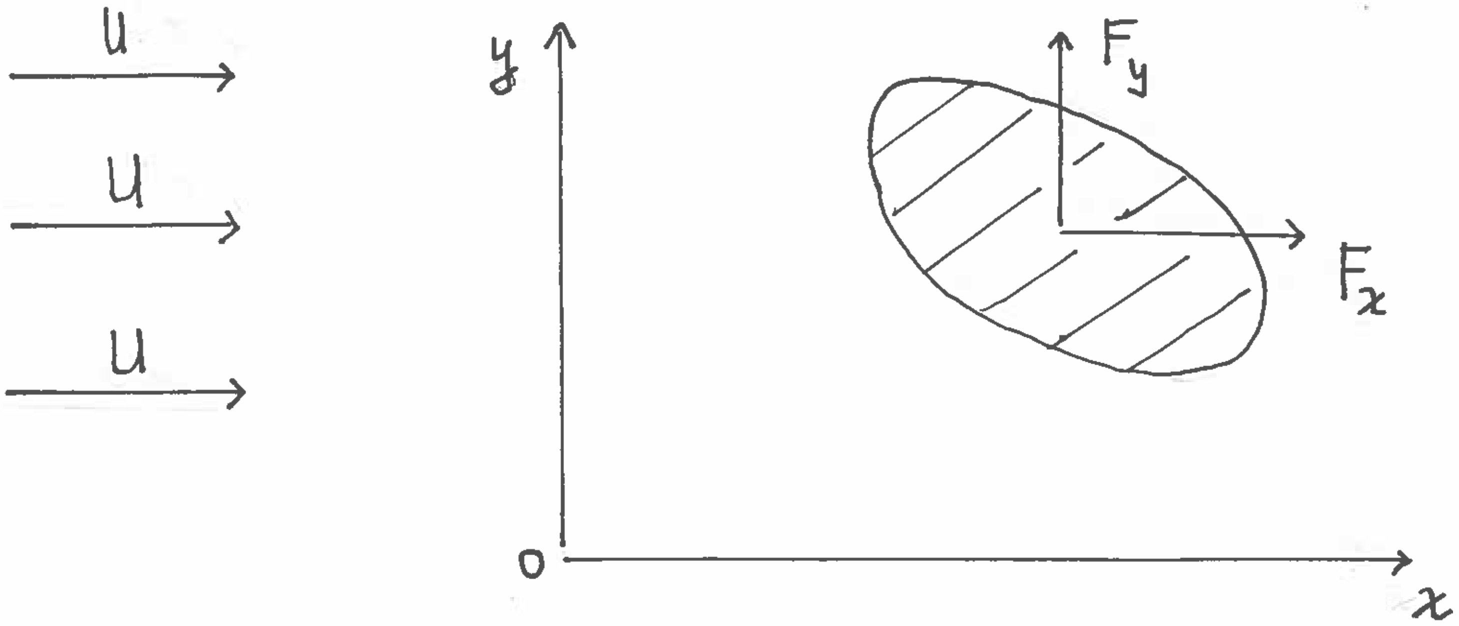
TWO-DIMENSIONAL
BODY



$$F_x - iF_y = \frac{1}{2} i\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz \qquad W(z) = \phi + i\psi$$

F_x AND F_y ARE THE COMPONENTS OF THE NET FORCE PER UNIT LENGTH ON THE BODY

- KUTTA-JOWOWSKI LIFT THEOREM



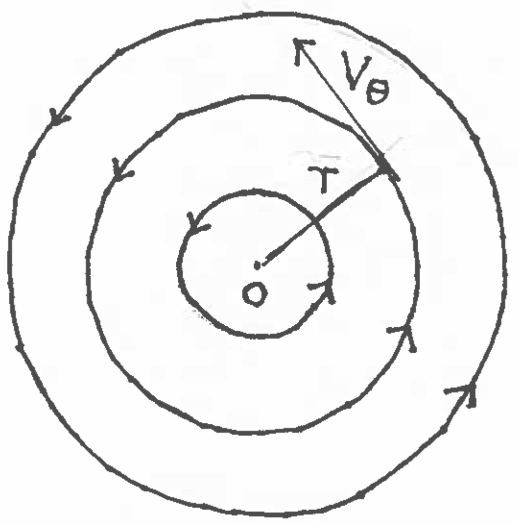
$$F_x = 0, \qquad F_y = -\rho U \Gamma$$

Γ = CIRCULATION ROUND BODY

FOR LIFT, $\Gamma < 0$

• VORTEX FLOW

ALL STREAMLINES ARE CONCENTRIC CIRCLES



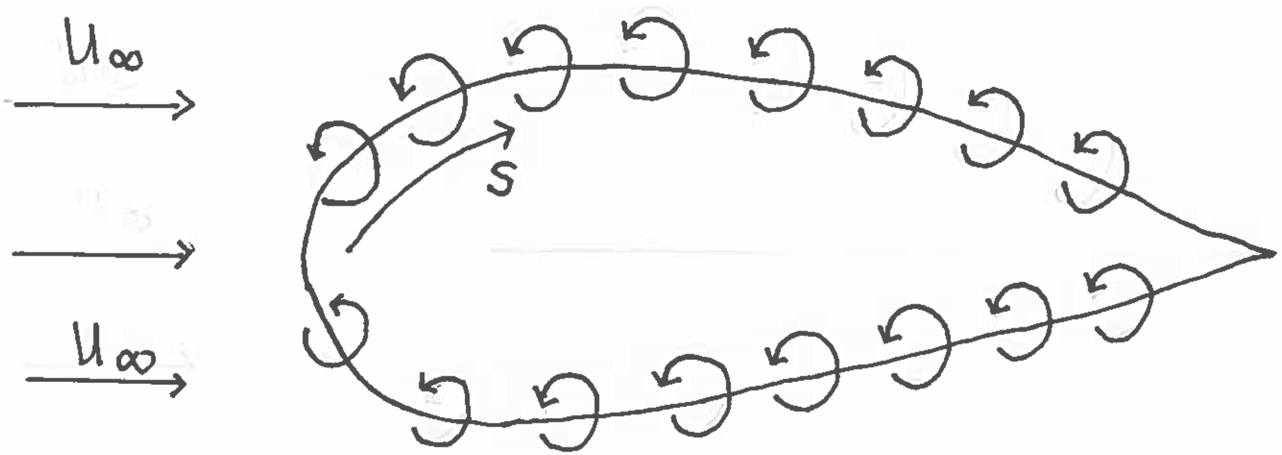
$$V_r = 0, \quad V_\theta = \frac{\Gamma}{2\pi r}, \quad V_z = 0$$

CIRCULATION Γ IS INDEPENDENT OF STREAMLINE

$$\underline{\nabla} \times \underline{V} = \underline{0} \quad \text{FOR } r \neq 0$$

Γ IS STRENGTH OF VORTEX FLOW

• VORTEX FILAMENT AND VORTEX SHEET



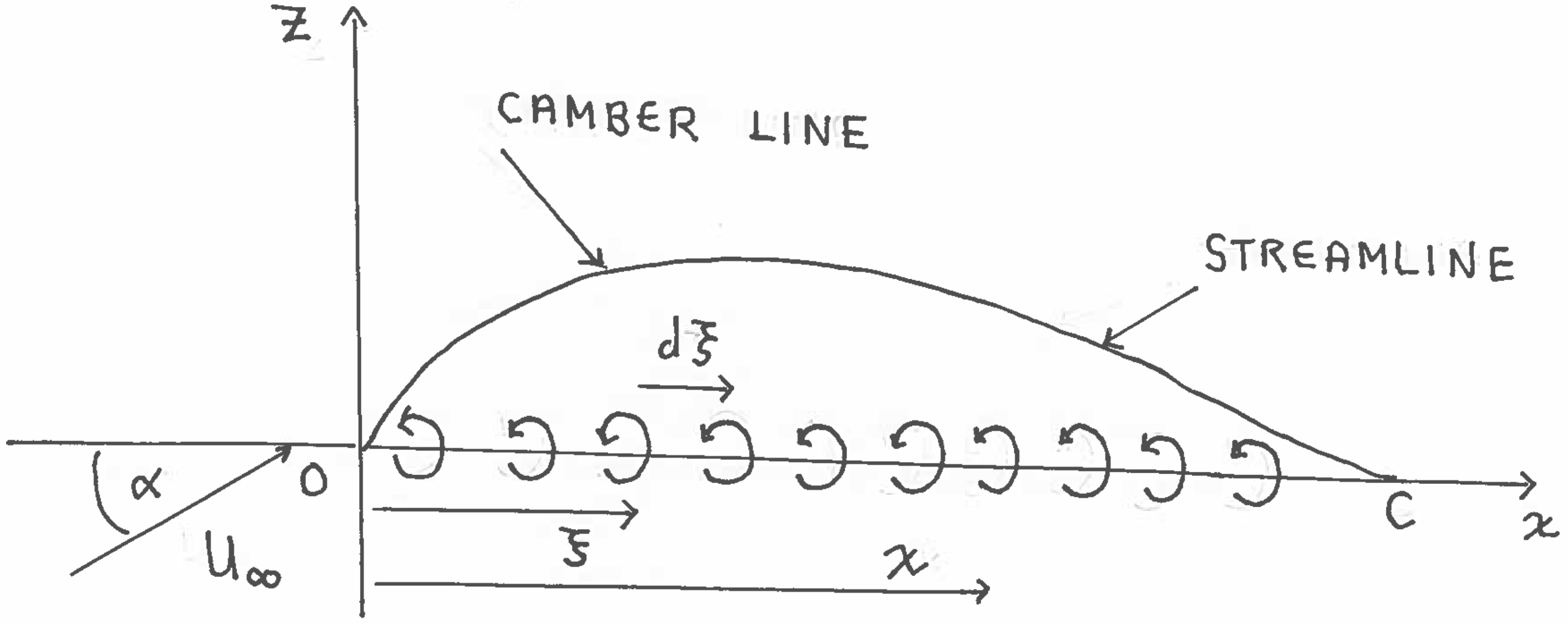
VORTEX SHEET DISTRIBUTED OVER AEROFOIL SURFACE

STRENGTH OF INFINITESIMAL ELEMENT OF SHEET = $\gamma_{(s)}\, ds$

- THIN AEROFOIL

PLACE VORTEX SHEET ON CHORD LINE

REQUIRE CAMBER LINE TO BE STREAMLINE OF FLOW



$$dW = \frac{\gamma(\xi)\, d\xi}{2\pi(\xi - x)}$$

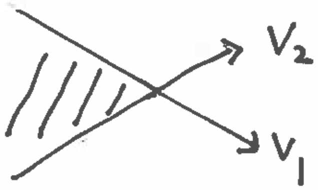
- AEROFOIL EQUATION

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)\,d\xi}{\xi - x} = \left(\alpha - \frac{dz}{dx}\right)U_\infty$$

- KUTTA CONDITION

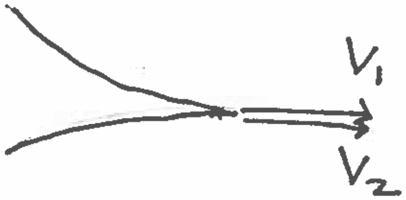
FOR STEADY FLOW OVER AN AEROFOIL AT GIVEN ANGLE OF ATTACK NATURE ADOPTS THE VALUE OF Γ WHICH RESULTS IN FLOW LEAVING SMOOTHLY THE TRAILING EDGE

FINITE ANGLE



$V_1 = V_2 = 0$

CUSP



$V_1 = V_2 \neq 0$

BOUNDARY CONDITION $\gamma(c) = 0$

3 LIFT GENERATION

AEROFOIL IS FIXED

- HOW DO THE BLADES IN A WIND TURBINE GENERATE LIFT
- HOW DOES A MOSQUITO GENERATE LIFT